On Connectivity

LI Qin

ABRIAL Jean-Raymond

April-May 2017

East China Normal University

A Bit of Recent History

- I just returned back from Shanghai.
- I spent two months at East China Normal University.
- I gave some lectures within the course of my colleague, Professor LI Qin.
- I covered some mathematical topics with Event-B and Rodin:
    - relations
    - well-foundedness
    - fixpoint
    - closure
    - connectivity

A Bit of Recent History (cont'd)

- I thought that the last topic on connectivity was very trivial.

- But I gradually figured out that it is not the case.

- I also worked with Professor LI Qin on his robotics project.

- In his project, he showed to me the importance of connectivity.

- We had to develop some mathematical properties of connectivity.

- Here is some ongoing work on this topic.

- Connecting relations on a set are quite useful in many occasions.

- Examples:
  - In network analysis (the relation between nodes in the network).
  - In robotic studies (the neighbourhood relation).

- In networks, it is important that nodes are connected so that communication is always possible between any two of them.

- In robotics, it is important that there are no groups of isolated robots in a population of little communicating neighbour robots.

Some References on Robotic Studies

Li Q, Smith G. Formal development of multi-agent systems using MAZE[J]. Science of Computer Programming, 2016
(This is the paper of Prof. LI Qin)

Stoy K. Using cellular automata and gradients to control self-reconfiguration[J]. Robotics & Autonomous Systems, 2006

Rubenstein M, Cornejo A, Nagpal R. Robotics. Programmable self-assembly in a thousand-robot swarm.[J]. Science, 2014

Outline

- Informal definition of connectivity.

- Three equivalent formal definitions of connectivity.

- Adding a point in a connected sets (while preserving connectivity).

- Removing a point from a connected set (without breaking connectivity).

Informal Definition of Connectivity

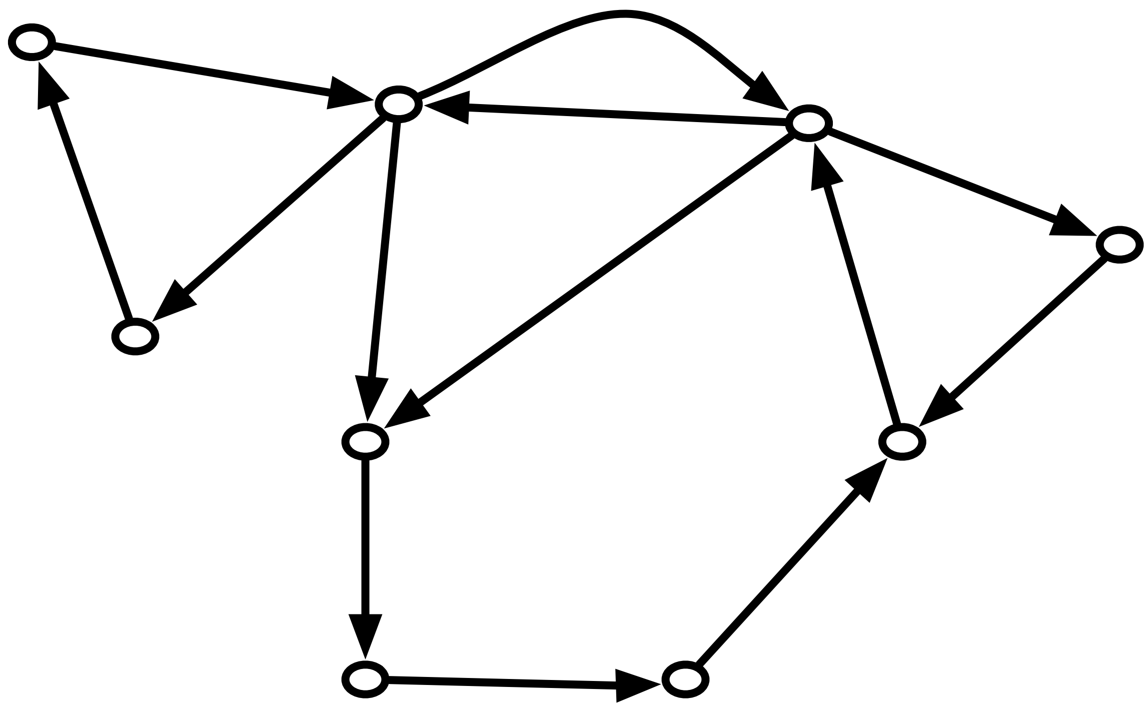
- Given a set A and a subset a of A.

- A relation r from a to a is said to connect the set a if ...

- ... any two points in a are directly or indirectly linked by r.

- Conditions on the relation r:
    - It is not empty: $r \neq \varnothing$.
    - Its domain is exactly a: $\mathbf{dom}(r) = a$.
    - It is irreflexive: $r \cap \mathbf{id} = \varnothing$.
    - In some cases, it is also symmetric: $r = r^{-1}$.

Example of a Connecting Relation



- All pair of points are directly or indirectly connected by the relation.

- Given a subset $\boldsymbol{a}$ of the set A, let $\boldsymbol{r}$ be the following relation:

$$\boldsymbol{r} \in \boldsymbol{a} \leftrightarrow \boldsymbol{a}$$

$$\boldsymbol{r} \neq \varnothing$$

$$\mathbf{dom}(\boldsymbol{r}) = \boldsymbol{a}$$

$$\boldsymbol{r} \cap \mathbf{id} = \varnothing$$

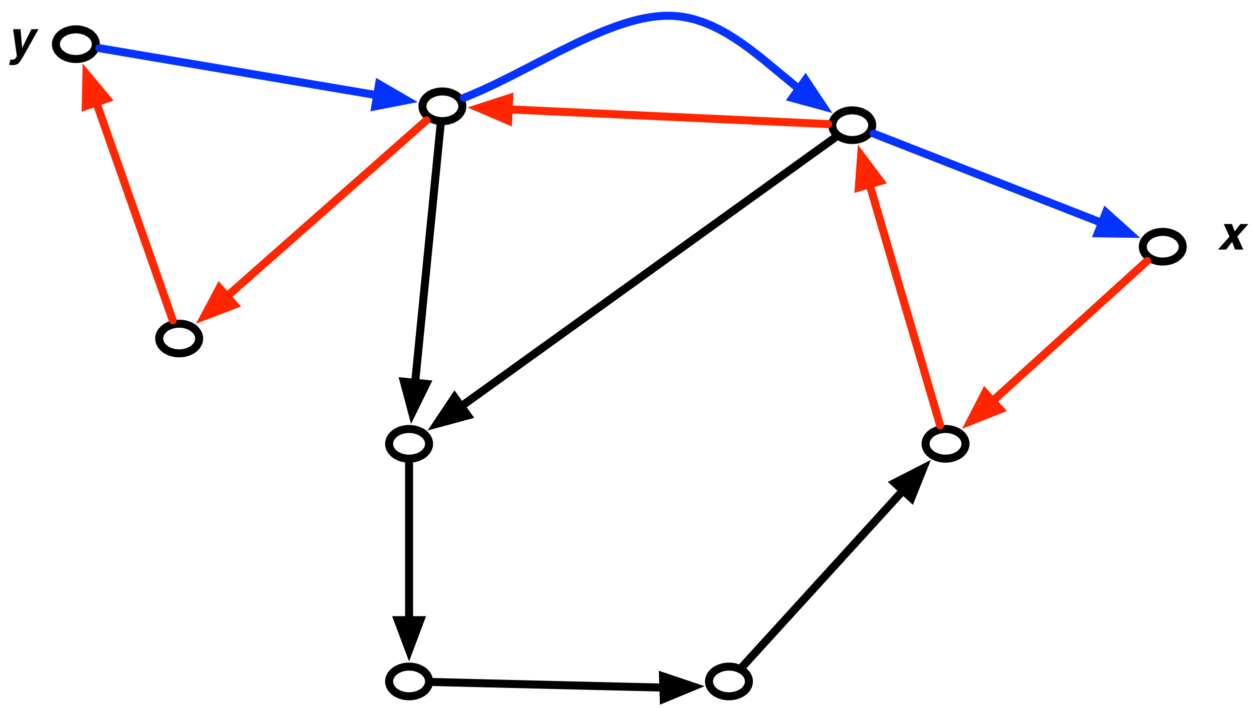
- Any two points are directly or indirectly connected by $\boldsymbol{r}$.

- They are thus connected by the irreflexive transitive closure $\boldsymbol{r}^+$:

$$\boldsymbol{a} \times \boldsymbol{a} \subseteq \boldsymbol{r}^+$$

- This is the first formal definition of connectivity.

- Connecting point x to point y and point y to point x:



A Second Formal Definition of Connectivity

- Again we have:

$$
\begin{array}{l}
r \in a \leftrightarrow a \\
r \neq \varnothing \\
\mathrm{dom}(r) = a \\
r \cap \mathrm{id} = \varnothing
\end{array}
$$

- Here is our second formal definition:

$$\forall s \cdot s \subseteq a \;\wedge\; s \neq \varnothing \;\wedge\; r[s] \subseteq s \;\Rightarrow\; a \subseteq s$$

- Nice to use it to prove inductive properties of a connected set.

- We can prove (with Rodin) the following equivalence :

$$a \times a \subseteq r^{+} \;\Leftrightarrow\; \forall s \cdot s \subseteq a \;\wedge\; s \neq \varnothing \;\wedge\; r[s] \subseteq s \;\Rightarrow\; a \subseteq s$$

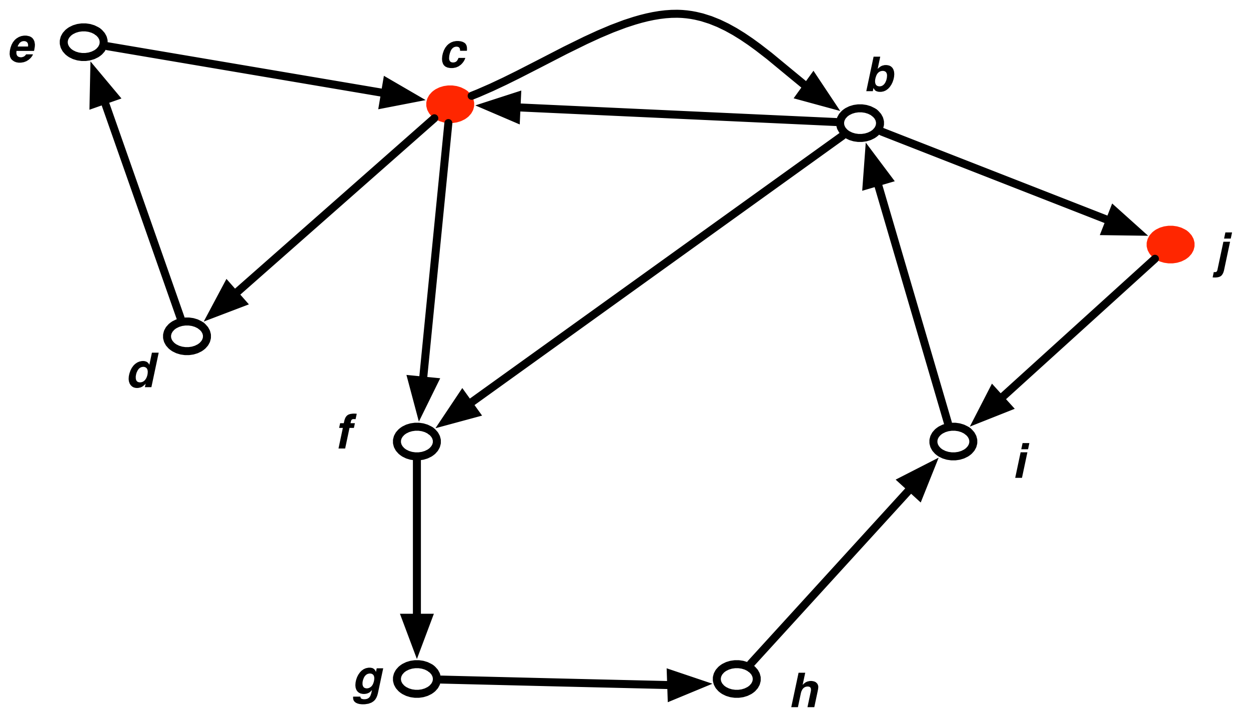
- Suppose that two points x and y of a are connected by relation r.

- Then, there are some sequences starting at x and ending at y where two successive points in these sequences are related by r.

- Here is the relation connecting two points to their sequences:

$$
\begin{array}{l}
path = \{x \mapsto y \mapsto p \mid x \in a \;\wedge\; y \in a \;\wedge \\
\qquad p \in \{s \mid \exists n \cdot n \in \mathbb{N} \;\wedge\; n > 1 \;\wedge \\
\qquad\qquad s \in 1\,..\,n \rightarrow a \;\wedge \\
\qquad\qquad s(1) = x \;\wedge\; s(n) = y \;\wedge \\
\qquad\qquad \forall i \cdot\; i \in 1\,..\,n-1 \\
\qquad\qquad\qquad \Rightarrow \\
\qquad\qquad\qquad s(i) \mapsto s(i+1) \in r \\
\qquad \} \\
\}
\end{array}
$$

Example



- Six sequences (and more) from point c to point j :

c, b, j $\qquad$ c, d, e, c, b, j $\qquad$ $\ldots$

c, f, g, h, i, b, j $\qquad$ $c, d, e, c, f, g, h, i, b, j$ $\qquad$ $\ldots$

c, b, f, g, h, i, b, j $\qquad$ $c, d, e, c, b, f, g, h, i, b, j$ $\qquad$ $\ldots$

A Third Formal Definition of Connectivity (cont'd)

- Any two points are connected by at least one of these sequences.

- Here is the third definition:

$$a \times a = \mathrm{dom}(path)$$

- We can prove (with Rodin) the following equivalence:

$$a \times a \subseteq r^{+} \;\Leftrightarrow\; a \times a = \mathrm{dom}(path)$$

Adding a Point to a Connected Set

- We want to preserve connectivity in the extended set.

- So, we suppose that the set a is already connected by the relation r.

- Let x be a point of A outside a.

- Let y and z be two points of a.

- We extend the relation r as follows:

$$r \cup \{x \mapsto z, y \mapsto x\}$$

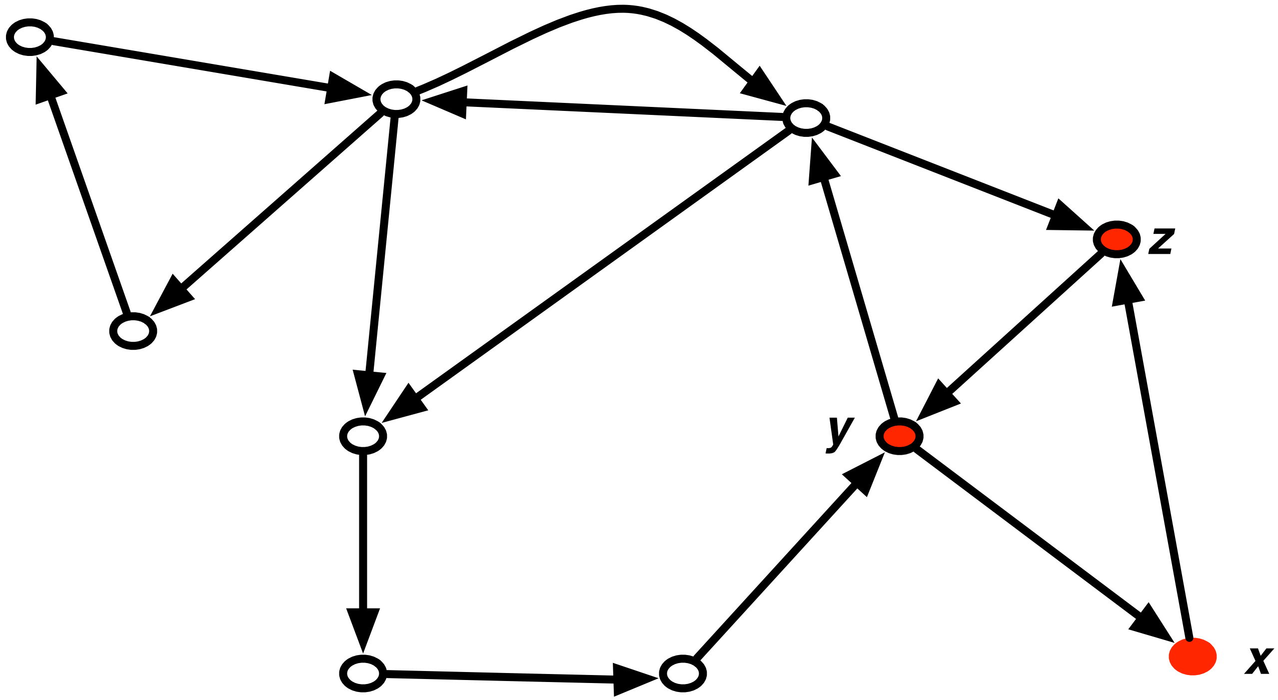
- We prove (with Rodin) that this new relation connects the set $a \cup \{x\}$:

$$\forall t \cdot t \subseteq a \;\wedge\; t \neq \varnothing \;\wedge\; r[t] \subseteq t \;\Rightarrow\; a \subseteq t$$

$$\Rightarrow$$

$$\forall t \cdot t \subseteq a \cup \{x\} \;\wedge\; t \neq \varnothing \;\wedge\; (r \cup \{x \mapsto z, y \mapsto x\})[t] \subseteq t \;\Rightarrow\; a \cup \{x\} \subseteq t$$

Example



- The point x is added and connected to points y and z.

- We want to do this without destroying connectivity.

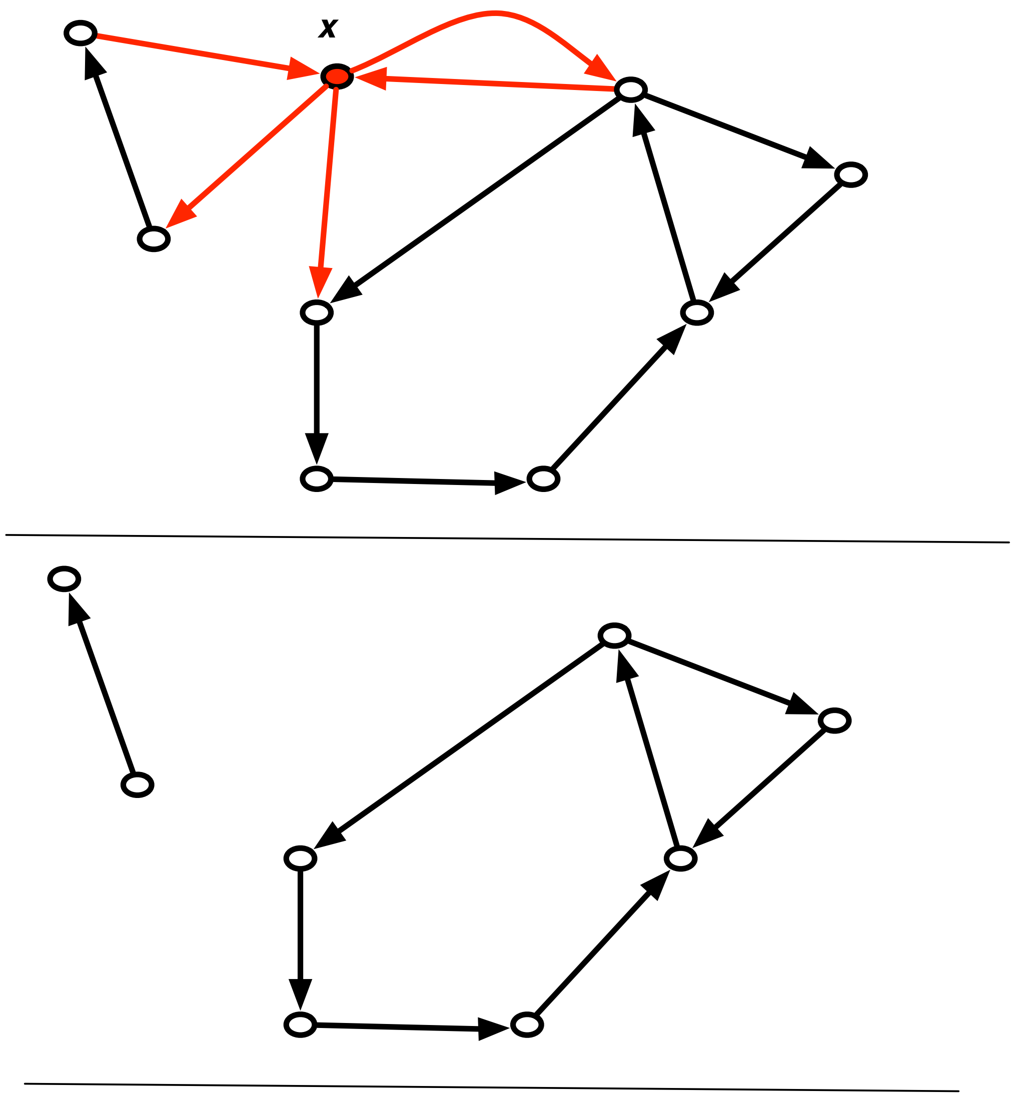
- This a challenging problem.

- We present a sufficient condition to remove a point.

- This is the "discrete distance algorithm" (Stoy, Rubenstein, Li, . . .)

- This algorithm is used and presented in some papers.

- To the best of our knowledge, this is done without any formal proof.



- Removing point x breaks connectivity.

Defining Proper Candidates for Removing

- A point z can be removed if for all pairs of distinct points x and y in $a \setminus \{z\}$ there exists a sequence linking x to y that does not contain z.

$$
\begin{array}{l}
candidate = \{z \mid z \in a \land \\
\qquad \forall x, y \cdot \; x \in a \setminus \{z\} \land \\
\qquad\qquad\quad y \in a \setminus \{z\} \land \\
\qquad\qquad\quad x \neq y \\
\qquad\qquad\quad \Rightarrow \\
\qquad\qquad\quad \exists p \cdot p \in path[\{x \mapsto y\}] \land z \notin \mathrm{ran}(p) \\
\qquad \}
\end{array}
$$

The set $candidate$ contains all points that can be removed

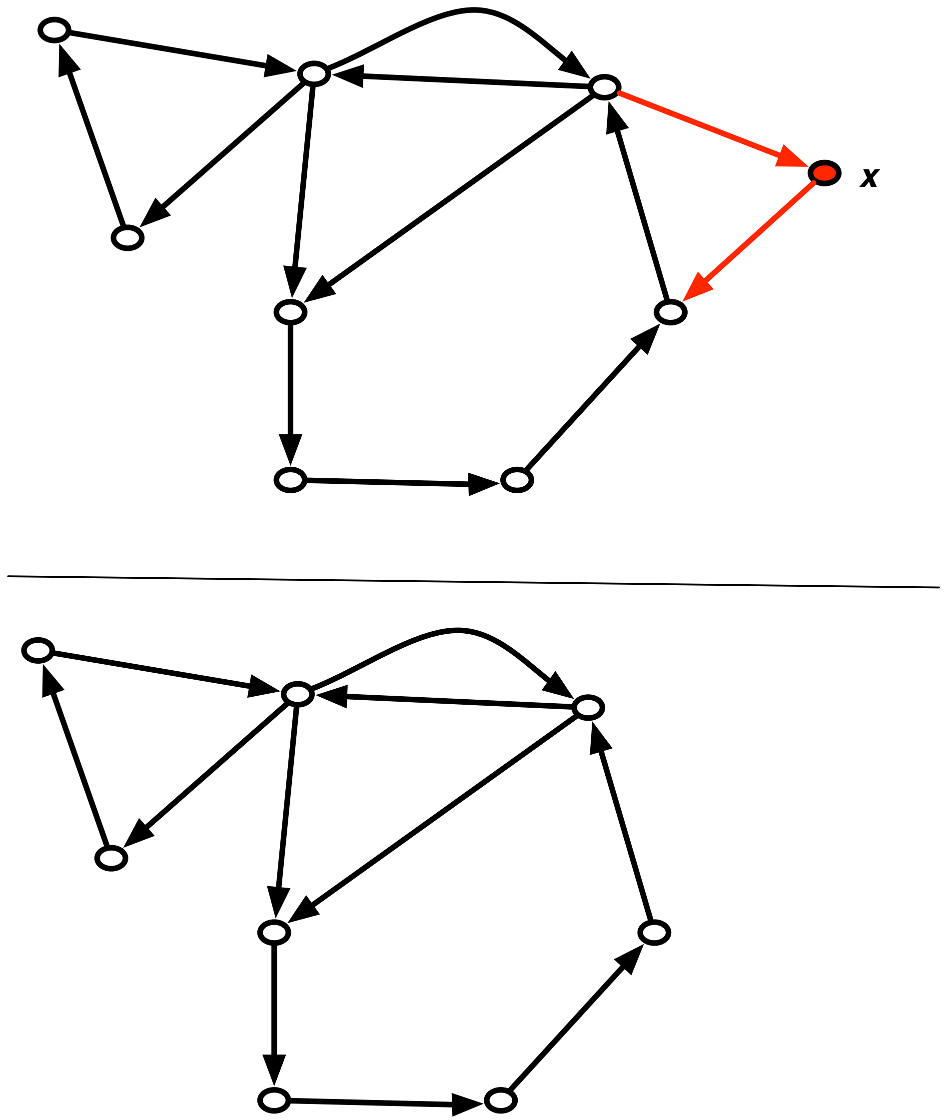
- Members of $candidate$ can be removed without breaking connectivity.

- We proved (with Rodin) the following:

$$
\begin{array}{l}
\forall u \cdot \; u \in candidate \\
\qquad \Rightarrow \\
\qquad (a \setminus \{u\} \times a \setminus \{u\}) \setminus \mathrm{id} \\
\qquad \subseteq \\
\qquad \{x \mapsto y \mapsto p \mid x \in a \setminus \{u\} \;\wedge\; y \in a \setminus \{u\} \;\wedge\; x \neq y \;\wedge \\
\qquad\qquad p \in \{s \mid \exists n \cdot n \in \mathbb{N} \;\wedge\; n > 1 \;\wedge \\
\qquad\qquad\qquad s \in 1\,..\,n \rightarrow a \;\wedge \\
\qquad\qquad\qquad s(1) = x \;\wedge\; s(n) = y \;\wedge \\
\qquad\qquad\qquad \forall i \cdot \; i \in 1\,..\,n-1 \\
\qquad\qquad\qquad\qquad \Rightarrow \\
\qquad\qquad\qquad\qquad s(i) \mapsto s(i+1) \in r \\
\qquad\qquad\} \\
\qquad \}
\end{array}
$$

- This means that the set $a \setminus \{u\}$ is still connected.

Example



- Removing point x does not break connectivity.

An Additional Assumption about the Connecting Relation

- This assumption is required by the "discrete distance algorithm".
- It is irreflexive.
- It is symmetric (this is the new assumption).
- It is not transitive.
- Example of the neighbourhood relation:
    - it is irreflexive (I am not my own neighbour),
    - it is symmetric (I am the neighbour of my neighbours),
    - It is not transitive (neighbours of my neighbours are not necessarily my neighbours).
- We do not want to break connectivity when removing one point.

Two Questions

1- Is $candidate$ always not empty for a connected set?

2 - How to determine that a point is indeed a member of $candidate$?

- The answer to 1 is "yes" provided the set has more than two points.

- The answer to 2 is the object of the "discrete distance algorithm".

- This algorithm requires an additional assumption on the relation r.

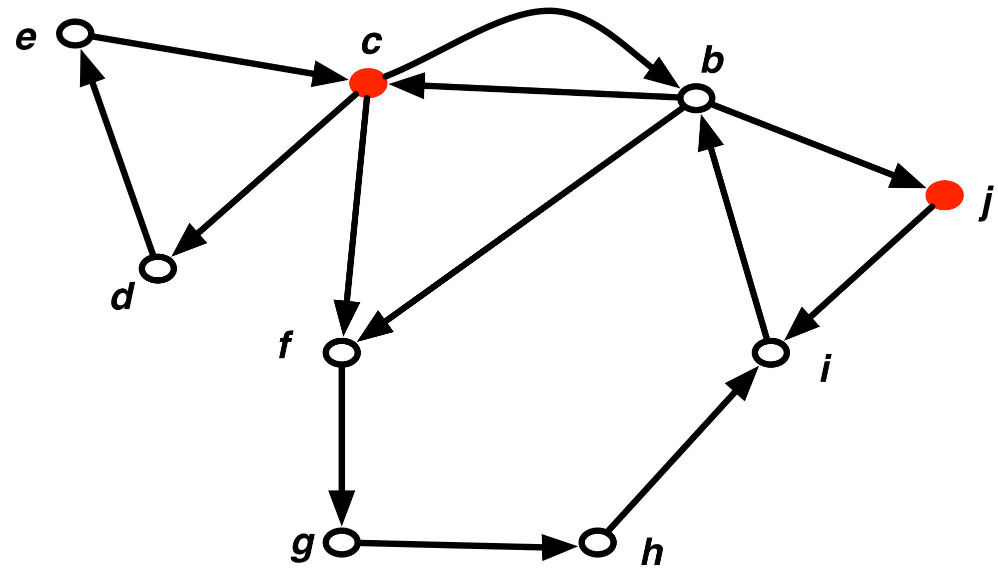
A Notion of Discrete Distance Between Pairs of Points

- We have seen that several sequences can link two different points x and y.

- We define the distance between x and y as the shortest length of such sequences minus 1:

$$dist(x \mapsto y) = \min(length[path[\{x \mapsto y\}]]) - 1$$

Example



- Six sequences (and more) from point c to point j :

$$c, b, j \qquad c, d, e, c, b, j \qquad \ldots$$

$$c, f, g, h, i, b, j \qquad c, d, e, c, f, g, h, i, b, j \qquad \ldots$$

$$c, b, f, g, h, i, b, j \qquad c, d, e, c, b, f, g, h, i, b, j \qquad \ldots$$

$$dist(c \mapsto j) = \min(\{3, 6, 7, 10, 11, \ldots\}) - 1 = 2$$

- We thus suppose that r is symmetric.

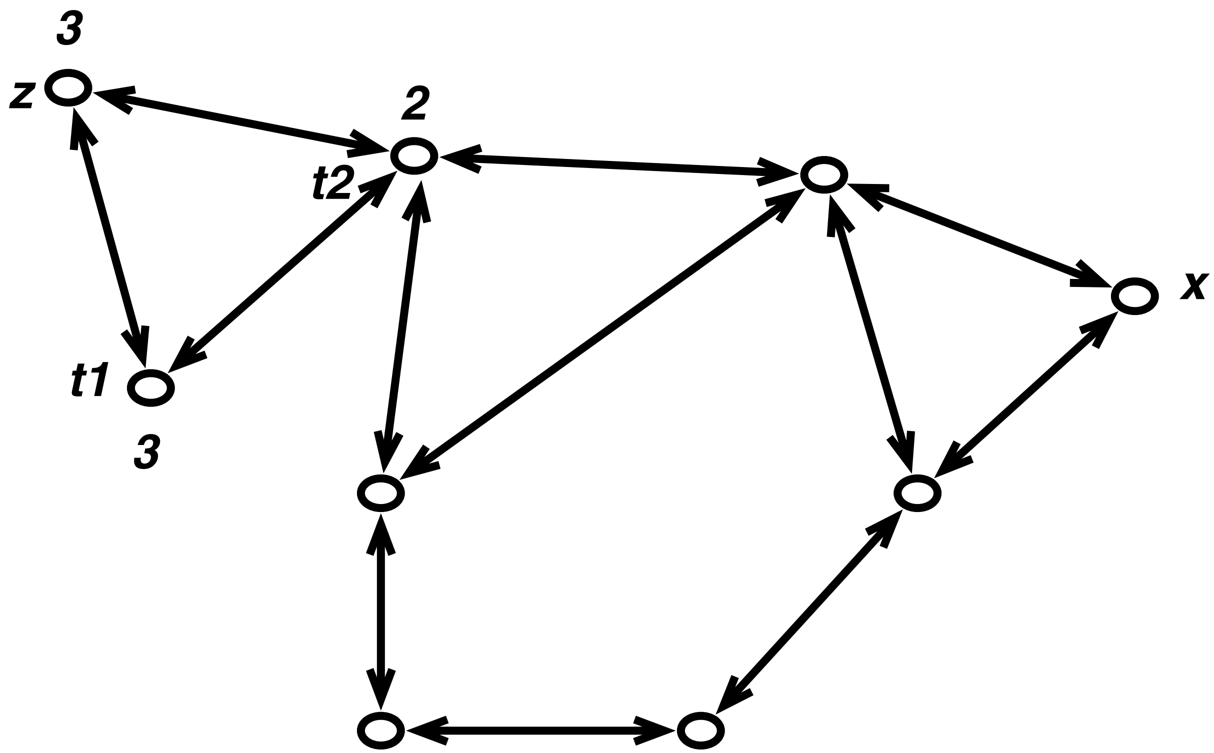
- Given a point x, the algorithm says the following.

- A point z, with a distance to x greater than or equal to the distance of his neighbours to x, is in $candidate$.

- We proved (with Rodin) the correctness of this algorithm:

$$
\begin{array}{ll}
\forall x, z \cdot & x \in a \;\land \\
& z \in a \;\land \\
& x \neq z \;\land \\
& \forall t \cdot t \in a \land z \mapsto t \in r \;\Rightarrow\; dist(z \mapsto x) \geq dist(t \mapsto x) \\
& \Rightarrow \\
& z \in candidate
\end{array}
$$

Example



- Point z can be removed

$$\begin{aligned} dist(z1 \mapsto x) &= 3 \\ dist(t1 \mapsto x) &= 3 \\ dist(t2 \mapsto x) &= 2 \end{aligned}$$

Future Work

- Writing a paper.

- Extending the slides with more on the "discrete distance algorithm".

- Simplifying the formal proofs done with Rodin (they are a bit heavy).

- The complete formal robot system has to be developed further.

- You can get slides and Rodin development: ask T.S. Hoang.

- A nice video: https://www.youtube.com/watch?v=JmyTJSYw77g

Thanks for listening