An Exercise in Mathematical Engineering:

Proving Weak and Strong Goodstein Theorems

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Mathematical Engineering: What does that means?

- We have always thought that math. and programming are related

- We have often observed that math. educated people are quite good

- But the vast majority of comp. professional is not math. educated

- How could we have more math. inserted in a CS curriculum?

- What kind of “mathematics” should we add to such a curriculum?

Mathematical Engineering (cont'd)

- The precise mathematical subject is not important

- What matters is the context of a mathematical subject

- "Context" is the background needed to formalise a math. subject:
    - definitions
    - axioms
    - imported and intermediate results
    - proofs, etc ...

- It is not so different from similar contexts encountered in software

- Why not incorporating such examples in CS curriculum?

Mathematical Engineering (cont'd)

- In order to make this idea more precise

- We started to dig into mathematical books and articles

- Our goal was to construct a data base of such math. contexts

- And then to try presenting these examples to some students

- In order to see how they react to this material

- BUT, we quickly discovered that ...

- Mathematical contexts (taken from books or articles) are often:
    - Badly structured.
    - Hard to understand.
    - With important definitions just missing.
    - Not abstract enough!!!

- Consequence: math. works (as such) are not good examples.

- We had no choice but to reconstruct some mathematical contexts.

- Here is such an example: the Goodstein theorem

- In 1944, Goodstein presented and proved a very strange result.

- Reference: Goodstein, R. (1944),
  "On the restricted ordinal theorem", Journal of Symbolic Logic.

- He proved that a certain sequence of numbers, that seems to
  increase extremely rapidly, is in fact not increasing for ever.

- Later, people simplified this result, thus introducing the, so called,
  "weak" Goodstein Theorem.

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1. Presentation of Goodstein Computations

Example of a Weak Goodstein Computation

- Given a natural number written in base 2: $\mathbf{2^8 + 2^3 + 2 = 266}$

- We use the same notation, now in base 3: $\mathbf{3^8 + 3^3 + 3 = 6,591}$

- And we subtract 1, yielding: $\mathbf{3^8 + 3^3 + 2 = 6,590}$

- We write this number in base 4: $\mathbf{4^8 + 4^3 + 2 = 65,602}$

- And we subtract 1, yielding $\mathbf{4^8 + 4^3 + 1 = 65,601}$

Example of a Weak Goodstein Computation (cont'd)

- We write this number in base 5: $\mathbf{5^8 + 5^3 + 1 = 390,751}$

- And we subtract 1, yielding $\mathbf{5^8 + 5^3 = 390,750}$

- We write this number in base 6 yielding: $\mathbf{6^8 + 6^3 = 1,679,832}$

- And we subtract 1, yielding $\mathbf{6^8 + 5.6^2 + 5.6 + 5 = 1,679,831}$

- And so on ...

$$\mathbf{266 \quad 6,590 \quad 65,601 \quad 390,750 \quad 1,679,831 \quad \ldots}$$

- It seems that this trace is going to increase for ever

- But after a gigantic increase, it will eventually decrease to 0

- This is what the weak Goodstein theorem says.

- How can we prove this?

- The computation is the same as that of the weak Goodstein
- Increasing the base and decreasing the result
- But one is not using the simple base decomposition any more
- Instead, one uses the hereditary base decomposition
- Example of such a decomposition:

$$\mathbf{266 = 1.2^{2^{2+1}} + 1.2^{2+1} + 1.2^{1}}$$

- Instead of the simple base decomposition:

$$\mathbf{266 = 1.2^{8} + 1.2^{3} + 1.2^{1}}$$

Example of a Strong Goodstein Trace

$$\begin{array}{lcl}
2^{2^{2+1}} + 2^{2+1} + 2 & = & 266 \\
3^{3^{3+1}} + 3^{3+1} + 3 - 1 & \approx & 4.4 \times 10^{36} \\
4^{4^{4+1}} + 4^{4+1} + 2 - 1 & \approx & 3.2 \times 10^{616} \\
5^{5^{5+1}} + 5^{5+1} + 1 - 1 & \approx & 2.5 \times 10^{1,0921} \\
6^{6^{6+1}} + 6^{6+1} - 1 & \approx & 3.5 \times 10^{217,832} \\
\ldots & &
\end{array}$$

- Again, it seems that this trace is going to increase for ever

- But after an extraordinary increase, it will eventually decrease to 0

- This is what the strong Goodstein theorem says

- How can we prove this?

2. Proof Approaches

Classical References

R.L. Goodstein *On the restricted ordinal theorem.* Journal of Symbolic Logic 9(1944)

L. Kirby and J. Paris *Accessible Independent Results for Peano Arithmetic.* Bulletin of the London Mathematical Society 4 (1982)

A. E. Caicedo *Goodstein's Theorem.* Revista Columbiana Matematicas (2007)

W. Sladek *The Termite and the Tower: Goodstein sequences and proverbiality in PA.* Draft (2007)

W. Gasarch *Theorems that you simply don't believe.* Computational Complexity Blog (2010)

M. Rathjen *Goodstein's Theorem Revisited* Draft (2014)

And many more . . .

Usual Approach to the Proof

- At each step in a Goodstein sequence one replaces the current base by ω, the smallest infinite ordinal. Example:

$$3^{3^{3+1}} + 3^{3+1} + 3 \qquad \text{is replaced by} \qquad \omega^{\omega^{\omega+1}} + \omega^{\omega+1} + \omega$$

- In doing so, we obtain an ordinal in, so called, Cantor Normal Form

- This set is known to be well-ordered

- Moreover, the -1 operation decreases it

- This is sufficient to prove Goodstein theorem

Issues we Found with Texts of this Proof in some Papers

- Ordinals do not form a set

- However ordinals from 0 to $\epsilon 0$ form a set: the Cantor normal form

- Where $\epsilon 0$ is such that $\epsilon 0 = \omega^{\epsilon 0}$ (ω exponentiated ω times)

- The -1 operation is not always defined for ordinals

- Not all presented proofs mentioned these issues

Our Approach of the Proof

- We found the proposed proof to be a bit magic.

- We would like to get rid of ordinals.

- We propose to replace the ordinals by some data structures.

- We want the strong Goodstein be a generalisation of weak one.

- We want to mechanise our proof with the Rodin Tool set.

- For this, we need some data structures for base notations

3. Data Structures for Base Notations

- We eliminate the base

- We replace the formula by a finite sequence

- Example for $\mathbf{266 = 2^8 + 2^3 + 2^1}$

$$\mathbf{= 1.2^8 + 1.2^3 + 1.2^1}$$

$$\mathbf{= 2^8.1 + 2^3.1 + 2^1.1}$$

- A finite sequence of pairs:

$$\mathbf{< (8 \mapsto 1), (3 \mapsto 1), (1 \mapsto 1) >}$$

Data Structure for Hereditary Base (1)

- We have a sequence of pairs as well

- But the first elements of these pairs are not numbers any more

- They are themselves sequence of pairs, and so on . . .

Data Structure for Hereditary Base (2)

- Example for $\mathbf{266 = 2^8 + 2^3 + 2^1}$

$$\mathbf{= 1.2^8 + 1.2^3 + 1.2^1}$$

$$\mathbf{= 1.2^{2^3} + 1.2^3 + 1.2^1}$$

$$\mathbf{= 1.2^{2^{2+1}} + 1.2^{2+1} + 1.2^1}$$

$$\mathbf{= 1.2^{1.2^{1.2^{1.2^0}+1.2^0}} + 1.2^{1.2^{1.2^0}+1.2^0} + 1.2^{1.2^0}}$$

$$\mathbf{= 2^{2^{2^{2^0.1}.1+2^0.1}.1}.1 + 2^{2^{2^0.1}.1+2^0.1}.1 + 2^{2^0.1}.1}$$

Data Structure for Hereditary Base (3)

- A finite sequence of pairs (finite sequence, number):

- Example for $266 = 2^{2^{2^{2^0.1}.1+2^0.1}.1}.1 + 2^{2^{2^0.1}.1+2^0.1}.1 + 2^{2^0.1}.1$

$$< (s1, 1), (s2, 1), (s3, 1) >$$

$$\begin{array}{lll} s1 & \leadsto & 2^{2^{2^{2^0.1}.1+2^0.1}.1} \\ s2 & \leadsto & 2^{2^{2^0.1}.1+2^0.1} \\ s3 & \leadsto & 2^{2^0.1} \end{array}$$

$$\begin{array}{lll} s1 & = & < (s2, 1) > \\ s2 & = & < (s3, 1), (<>, 1) > \\ s3 & = & < (<>, 1) > \end{array}$$

$$\begin{array}{lll} s1 & = & < (< (< (<>, 1) >, 1), (<>, 1) >, 1) > \\ s2 & = & < (< (<>, 1) >, 1), (<>, 1) > \\ s3 & = & < (<>, 1) > \end{array}$$

4. More on our approach of the proof

Some Ideas for the Proof

- Given the previously mentioned set S of data structures
- We want first to prove that S is well-ordered by a relation $\prec$
- Each number n with a base b can be transformed into an element g of S
- Note that incrementing the base b, by moving n into m, does not modify g
- Let h be the transformation of $m - 1$ with base $b + 1$ into a member of S
- We want then to prove the following: $h \prec g$
- Then, since S is well-ordered by $\prec$, THIS CANNOT BE DONE FOR EVER
- This approach will be used for both weak and strong Goodstein theorems

5. Some Basic Results

Let S be a set strictly well-ordered by a relation $\prec$.

Theorem 1: *The set $S \times \mathbb{N}1$ is strictly well-ordered by lexicographical ordering built with $\prec$ and $<$.*

Theorem 2: *The set of decreasing finite sequences built on S is strictly well-ordered by lexicographical ordering.*

Theorem 3: *Given two positive natural numbers x and n, we have:*

$$x^n - 1 \; = \; (x - 1) . \sum_{i=0}^{n-1} x^i$$

Example for Theorem 1

- Let S be a set strictly well-ordered by a relation $\prec$.

Theorem 1: *The set $S \times \mathbb{N}\mathbf{1}$ is strictly well-ordered by lexicographical ordering built with $\prec$ and $<$.*

- Let S be the set $\mathbf{0} \mathinner{\ldotp\ldotp} \mathbf{9}$ and let us reduce the set $\mathbb{N}\mathbf{1}$ to $\mathbf{1} \mathinner{\ldotp\ldotp} \mathbf{9}$

- The set $\{(\mathbf{0}, \mathbf{1}), (\mathbf{0}, \mathbf{2}), \ldots, (\mathbf{9}, \mathbf{9})\}$ is clearly lexicographically well-ordered

Example for Theorem 2

- Let S be a set strictly well-ordered by a relation $\prec$.

Theorem 2: *The set of decreasing finite sequences built on S is strictly well-ordered by lexicographical ordering.*

- Let S be the set $\mathbb{N}$ of natural numbers

- Then the following set of decreasing sequences is well-ordered:

$$\{< \mathbf{35}, \mathbf{22}, \mathbf{19}, \mathbf{11}, \mathbf{7} >, < \mathbf{35}, \mathbf{22}, \mathbf{20}, \mathbf{9}, \mathbf{3}, \mathbf{1} >, < \mathbf{38}, \mathbf{15} >, \ldots\}$$

Theorem 3: *Given two positive natural numbers x and n, we have:*

$$x^n - 1 = (x-1).\sum_{i=0}^{n-1} x^i$$

- Here is an example:

$$\begin{aligned} 6^3 - 1 &= 216 - 1 = 215 \\ &= 5.6^2 + 5.6 + 5 = 180 + 30 + 5 = 215 \end{aligned}$$

6. Properties of the data structures

- We have a set of sequences built on the set of pairs $\mathbb{N} \times \mathbb{N}\mathbf{1}$

- According to Theorem 1, this set is lexicographically ordered

- Moreover, the first element of the pair is decreasing

- Example:

$$< (\mathbf{8} \mapsto \mathbf{1}), (\mathbf{3} \mapsto \mathbf{1}), (\mathbf{1} \mapsto \mathbf{1}) >$$

- Thus, according to Theorem 2, this set is also well ordered

Theorem 4: *The set of data structures associated with simple bases is strictly lexicographically well-ordered*

- Let $\boldsymbol{T}$ be the set of data structure

- $\boldsymbol{T}$ can be inductively built from the following fixpoint equation:

$$\boldsymbol{T} = \mathsf{seq}(\boldsymbol{T} \times \mathbb{N}\mathbf{1})$$

- Example for $\mathbf{266 = 1.2^{2^{2+1}} + 1.2^{2+1} + 1.2^{1}}$

- A finite sequence of pairs:

$$\boldsymbol{< (s1 \mapsto 1), (s2 \mapsto 1), (s3 \mapsto 1) >}$$

$$\begin{aligned} \boldsymbol{s1} &\boldsymbol{=< (s2, 1) >} \\ \boldsymbol{s2} &\boldsymbol{=< (s3, 1), (<>, 1) >} \\ \boldsymbol{s3} &\boldsymbol{=< (<>, 1) >} \end{aligned}$$

- **Theorem 5**: *The set T is strictly and totally lexicographically ordered by means of a relation denoted by $\prec$.*

- Let LOD be the subset of T where each sequence, which is an element of T, is supposed to be decreasing along $\prec$.

- Each element of the set LOD has a height defined recursively on the structure of LOD

- **Theorem 6**: *Given two elements $s1$ and $s2$ of LOD with respective heights $h(s1)$ and $h(s2)$, we have:*

$$h(s1) < h(s2) \Rightarrow s1 \prec s2$$

- **Theorem 7**: *Every non-empty (and potentially infinite) subset of LOD, containing only elements with a height that is smaller than or equal to a certain height h, has a smallest element*

- **Theorem 8**: *The set LOD is lexicographically well-ordered by the relation $\prec$.*

7. Value Associated with a Base and a Data Structure

Simple Base Notation

$$\mathsf{vals}_b \in \mathsf{seq}(\mathbb{N} \times \mathbb{N}\mathbf{1}) \rightarrow \mathbb{N}$$

$$\begin{aligned} \mathsf{vals}_b(s \leftarrow (e, c)) &= \mathsf{vals}_b(s) + c.b^e \\ \mathsf{vals}_b(<>) &= 0 \end{aligned}$$

- Example:

$$\begin{aligned} \mathsf{vals}_2(<(8,1),(3,1),(1,1)>) &= 1.2^8 + 1.2^3 + 1.2^1 \\ &= 266 \end{aligned}$$

Hereditary Base Notation (Definition)

$$\mathsf{valt}_{\boldsymbol{b}} \in \mathsf{seq}(\boldsymbol{LOD}) \to \mathbb{N}$$

$$\begin{array}{lcl}
\mathsf{valt}_{\boldsymbol{b}}(\boldsymbol{s} \leftarrow (\boldsymbol{t}, \boldsymbol{c})) & = & \mathsf{valt}_{\boldsymbol{b}}(\boldsymbol{s}) + \boldsymbol{c}.\boldsymbol{b}^{\boldsymbol{valt_b(t)}} \\
\mathsf{valt}_{\boldsymbol{b}}(<>) & = & \mathbf{0}
\end{array}$$

Hereditary Base Notation (Example)

$$\begin{aligned}
\mathsf{valt}_2(s3) &= \mathsf{valt}_2(<(<>,1)>) \\
&= 1.2^{\mathsf{valt}_2(<>)} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\mathsf{valt}_2(s2) &= \mathsf{valt}_2(<(s3,1),(<>,1)>) \\
&= 1.2^{\mathsf{valt}_2(s3)} + 1.2^{\mathsf{valt}_2(<>)} \\
&= 2+1
\end{aligned}$$

$$\begin{aligned}
\mathsf{valt}_2(s1) &= \mathsf{valt}_2(<(s2,1)>) \\
&= 1.2^{\mathsf{valt}_2(s2)} \\
&= 2^{2+1}
\end{aligned}$$

$$\begin{aligned}
&\mathsf{valt}_2(<(s1,1),(s2,1),(s3,1)>) \\
&\quad = 1.2^{\mathsf{valt}_2(s1)} + 1.2^{\mathsf{valt}_2(s2)} + 1.2^{\mathsf{valt}_2(s3)} \\
&\quad = 2^{2^{2+1}} + 2^{2+1} + 2^1 \\
&\quad = 266
\end{aligned}$$

8. Data Structure Associated with a Base and a Number

Simple Base Notation (Definition)

$$\mathsf{seqs}_b \in \mathbb{N} \to \mathsf{seq}(\mathbb{N} \times \mathbb{N}1)$$

$$\mathsf{seqs}_b(n) \quad = \mathsf{seqs1}_b(0, n)$$

$$\mathsf{seqs1}_b(i, n) =$$

$$\begin{cases} \mathsf{seqs1}_b(i+1, n \;\mathsf{div}\; b) \leftarrow (i, n \;\mathsf{mod}\; b & \\ \qquad\qquad\qquad\qquad \text{if} \quad n \geq b \;\wedge\; n \;\mathsf{mod}\; b \neq 0 \\ \mathsf{seqs1}_b(i+1, n \;\mathsf{div}\; b) & \text{if} \quad n \geq b \;\wedge\; n \;\mathsf{mod}\; b = 0 \\ <(i, n)> & \text{if} \quad n < b \;\wedge\; n > 0 \\ <> & \text{if} \quad n = 0 \end{cases}$$

Simple Base Notation (Example)

$$
\begin{aligned}
\mathsf{seqs}_2(266) &= \mathsf{seqs1}_2(0, 266) \\
\mathsf{seqs1}_2(0, 266) &= \mathsf{seqs1}_2(1, 133) \\
&= \mathsf{seqs1}_2(2, 66) \leftarrow (1, 1) \\
&= \mathsf{seqs1}_2(3, 33) \leftarrow (1, 1) \\
&= \mathsf{seqs1}_2(4, 16) \leftarrow (3, 1) \leftarrow (1, 1) \\
&= \ldots \\
&= \mathsf{seqs1}_2(8, 1) \leftarrow (3, 1) \leftarrow (1, 1) \\
&= <(8, 1)> \leftarrow (3, 1) \leftarrow (1, 1) \\
&= <(8, 1), (3, 1), (1, 1)>
\end{aligned}
$$

Theorem 9:

$$\forall n, b \cdot n > 0 \;\wedge\; b > 1 \;\Rightarrow\; \mathsf{seqs}(b)(n-1) \prec \mathsf{seqs}(b)(n)$$

Theorem 10:

$$\begin{aligned}\forall n, b, B \cdot\; & n \in \mathbb{N} \;\wedge \\ & b > 1 \;\wedge \\ & B \geq b \\ & \Rightarrow \\ & \mathsf{seqs}(B)(\mathsf{vals}(B)(\mathsf{seqs}(b)(n))) = \mathsf{seqs}(b)(n)\end{aligned}$$

Theorem 11:

$$\begin{aligned}\forall n, b, B \cdot\; & n \in \mathbb{N} \;\wedge \\ & b > 1 \;\wedge \\ & B \geq b \\ & \Rightarrow \\ & \mathsf{seqs}(B)(\mathsf{vals}(B)(\mathsf{seqs}_b(n)) - 1) \prec \mathsf{seqs}(b)(n)\end{aligned}$$

$$\mathsf{seqt}_b \in \mathbb{N} \to \mathsf{seq}(LOD)$$

$$\mathsf{seqt}_b(n) \quad = \mathsf{seqt1}_b(0, n)$$

$$\mathsf{seqt1}_b(i, n) =$$

$$\begin{cases} \mathsf{seqt1}_b(i+1, n \;\mathsf{div}\; b) \leftarrow (\mathsf{seqt1}_b(0, i), n \;\mathsf{mod}\; b) & \\ \qquad \text{if} \quad n \geq b \;\wedge\; n \;\mathsf{mod}\; b \neq 0 \\ \mathsf{seqt1}_b(i+1, n \;\mathsf{div}\; b) & \text{if} \quad n \geq b \;\wedge\; n \;\mathsf{mod}\; b = 0 \\ < (\mathsf{seqt1}_b(0, i), n) > & \text{if} \quad n < b \;\wedge\; n > 0 \\ <> & \text{if} \quad n = 0 \end{cases}$$

Hereditary Notation (Example)

$\text{seqt}_2(266) = \text{seqt1}_2(0, 266)$

$\text{seqt1}_2(0, 266)$

$= \text{seqt1}_2(1, 133)$

$= \text{seqs1}_2(2, 66) \leftarrow (\text{seqt1}_2(0, 1), 1)$

$= \text{seqs1}_2(3, 33) \leftarrow (\text{seqt1}_2(0, 1), 1)$

$= \text{seqs1}_2(4, 16) \leftarrow (\text{seqt1}_2(0, 3), 1) \leftarrow (\text{seqt1}_2(0, 1), 1)$

$= \ldots$

$= \text{seqs1}_2(8, 1) \leftarrow (\text{seqt1}_2(0, 3), 1) \leftarrow (\text{seqt1}_2(0, 1), 1)$

$=< (\text{seqt1}_2(0, 8), 1) > \leftarrow (\text{seqt1}_2(0, 3), 1) \leftarrow (\text{seqt1}_2(0, 1), 1)$

$=< (\text{seqt1}_2(0, 8), 1), (\text{seqt1}_2(0, 3), 1), (\text{seqt1}_2(0, 1), 1) >$

$=< (s1, 1), (s2, 1), (s3, 1) >$

$$
\begin{aligned}
s3 &= (\mathsf{seqt1}_2(0,1) \\
&= <\ (\mathsf{seqt1}_2(0,0),1)\ > \\
&= <\ (<>,1)\ >
\end{aligned}
$$

$$
\begin{aligned}
s2 &= \mathsf{seqt1}_2(0,3) \\
&= \mathsf{seqt1}_2(1,1) \leftarrow (\mathsf{seqt1}_2(0,0),1) \\
&= \mathsf{seqt1}_2(1,1) \leftarrow s3 \\
&= <\ (\mathsf{seqt1}_2(0,1),1)\ >\leftarrow s3 \\
&= <\ (s3,1),(<>,1)\ >
\end{aligned}
$$

$$
\begin{aligned}
s1 &= \mathsf{seqt1}_2(0,8) \\
&= \mathsf{seqt1}_2(1,4) \\
&= \mathsf{seqt1}_2(2,2) \\
&= \mathsf{seqt1}_2(3,1) \\
&= <\ (\mathsf{seqt1}_2(0,3),1)\ > \\
&= <\ (s2,1)\ >
\end{aligned}
$$

Theorem 12: *The range of the function* seqt(b) *is included in the set* LOD

Theorem 13:
$\forall n, b \cdot n > 0 \ \land \ b > 1 \ \Rightarrow \ \text{seqt}(b)(n-1) \prec \text{seqt}(b)(n)$

Theorem 14:

$$\begin{array}{ll} \forall n, b, B \cdot & n \in \mathbb{N} \ \land \\ & b > 1 \ \land \\ & B \geq b \\ & \Rightarrow \\ & \text{seqt}(B)(\text{valt}(B)(\text{seqt}(b)(n))) = \text{seqt}(b)(n) \end{array}$$

Theorem 15:

$$\begin{array}{ll} \forall n, b, B \cdot & n \in \mathbb{N} \ \land \\ & b > 1 \ \land \\ & B \geq b \\ & \Rightarrow \\ & \text{seqt}(B)(\text{valt}(B)(\text{seqt}(b)(n)) - 1) \prec \text{seqt}(b)(n) \end{array}$$

9. Goodstein Proofs

- We have to prove the termination of this loop

$n :=$ some natural number;
$b := 2;$
while $n \neq 0$ **do**
  $n := \text{vals}_{b+1}(\text{seqs}_b(n)) - 1;$
  $b := b + 1$
end

- The state of this program is the pair $b \mapsto n$

- We have to define a "variant" and prove that it decreases

- Our candidate is $\text{seqs}(b)(n)$

- We have to prove the decreasing:

$$\text{seqs}(b+1)(\text{vals}(b+1)(\text{seqs}(b)(n)) - 1) \prec \text{seqs}(b)(n)$$

where $\prec$ denotes the lexicographical order built on finite sequences of $\mathbb{N} \times \mathbb{N}\mathbf{1}$.

Theorem 16 (Weak Goodstein theorem): *The previous loop terminates*

- We have to prove the termination of this loop

$n :=$ some natural number;
$b := 2;$
while $n \neq 0$ **do**
  $n := \mathrm{valt}_{b+1}(\mathrm{seqt}_b(n)) - 1;$
  $b := b + 1$
end

- The state of this program is the pair $b \mapsto n$

- We have to define a "variant" and prove that it decreases

- Our candidate is seqt$(b)(n)$

- We have to prove the decreasing:

$$\text{seqt}(b+1)(\text{valt}(b+1)(\text{seqt}(b)(n))-1) \prec \text{seqt}(b)(n)$$

where $\prec$ denotes the lexicographical order built on finite sequences of LOD.

Theorem 17 (Strong Goodstein theorem): *The previous loop terminates*

10. Discussion and Conclusion

Back to the Usual Proof

- At each step in a Goodstein sequence one replaces the current base by ω, the smallest infinite ordinal. Example:

$$3^{3^{3+1}} + 3^{3+1} + 3 \qquad \text{is replaced by} \qquad \omega^{\omega^{\omega+1}} + \omega^{\omega+1} + \omega$$

- In doing so, we obtain an ordinal in, so called, Cantor Normal Form

- This set is well-ordered and the -1 operation decreases it

- This is sufficient to prove Goodstein theorem

What we Have Done

- The usual proof reasons on Cantor normal form

- Our set LOD is an encoding of Cantor normal form

- We have not used transfinite numbers, but they were not very far

Thanks for Listening